LOCALLY POTENTIALLY EQUIVALENT GALOIS REPRESENTATIONS

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ABSTRACT. We show that if two continuous semi-simple ℓ-adic Galois representations are locally potentially equivalent at a sufficiently large set of places then they are globaly potentially equivalent. We also prove an analogous result for arbitrarily varying powers of character values evaluated at the Frobenius conjugacy classes. In the context of modular forms, we prove: given two non-CM newforms f and g of weight at least two, such that $a_p(f)^{n_p} = a_p(g)^{n_p}$ on a set of primes of positive upper density and for some set of natural numbers n_p, then f and g are twists of each other by a Dirichlet character.

1. INTRODUCTION

Let $S(k, N, \epsilon)$ be the collection of newforms on $\Gamma_1(N)$ of weight k and Nebentypus character ϵ. Given $f \in S(k, N, \epsilon)$ and p a rational prime coprime to N, let $a_p(f)$ be the eigenvalue of f with respect to the Hecke operator at p. An initial motivation for this paper is to prove the following type of recovery theorem with varying local conditions:

Theorem 1.1. *Let $f_i \in S(k_i, N_i, \epsilon_i)$, $i = 1$, 2 be two newforms, one of them a non-CM form of weight at least two. Suppose that for a collection T of rational primes p coprime to $N_1 N_2$ of positive upper density, there exists natural numbers n_p for $p \in T$ such that*

$$a_p(f_1)^{n_p} = a_p(f_2)^{n_p}, \ \forall \ p \in T.$$

Then there exists a Dirichlet character χ such that

$$f_2 \simeq f_1 \otimes \chi,$$

i.e., for all p coprime to $N_1 N_2$,

$$a_p(f_2) = a_p(f_1)\chi(p).$$

When $n_p = n$ is a constant and the density of the set of places T is one, the above theorem is proved in [Ra2].

In this note, we look at these questions in the general context of ℓ-adic semisimple representations of the absolute Galois groups of a global field K. In this context, the generalization can be viewed from two different perspectives. We show that ℓ-adic semisimple representations of the absolute Galois groups of a global field K which are

locally potentially equivalent at a sufficiently large set of places K are in fact globally potentially equivalent. On the other hand, we can consider an equality of arbitrarily varying powers of the character values evaluated at the Frobenius conjugacy classes at a sufficiently large set of places, and deduce potential equivalence of the global representations.

The method of proof is as in [Se, Ra1], to look at the algebraic monodromy groups attached to the Galois representations, and to use algebraic versions of the Chebotarev density theorem. This allows the use of algebraic and transcendental methods, for example the orthogonality relation for characters of compact groups, to arrive at the desired conclusion.

2. POTENTIALLY EQUIVALENT GALOIS REPRESENTATIONS

Let K be a global field and let $G_K = \mathrm{Gal}(\bar{K}/K)$ denote the absolute Galois group over K of a separable closure $\bar{K}$ of K. Let ℓ be a rational prime coprime to the characteristic of K and let F be a ℓ-adic local field of characteristic zero. Suppose $\rho : G_K \to GL_n(F)$ is a continuous semisimple representation of G_K. We will assume that ρ is unramified outside a finite set of places of K. At a place v of K where ρ is unramified, let σ_v denote the Frobenius conjugacy class in the quotient group $G_K/\mathrm{Ker}(\rho)$. By an abuse of notation, we will also continue to denote by σ_v an element in the associated conjugacy class.

For any place v of K, let K_v denote the completion of K at v, and G_{K_v} the corresponding local Galois group. Choosing a place w of $\bar{K}$ lying above v, allows us to identity G_{K_v} with the decomposition subgroup D_w of G_K. As w varies this gives a conjugacy class of subgroups of G_K. Given a representation ρ of G_K as above, define the localization (or the local component) ρ_v of ρ at v, to be the representation of G_{K_v} obtained by restricting ρ to a decomposition subgroup. This is well defined upto isomorphism.

For a finite place v of K, denote by Nv the cardinality of the residue field of K_v. The upper density of a set S of finite places of K is defined as:

$$ud(S) := \limsup_{x \to \infty} \#\{v \in S|\ Nv \le x\}/\pi(x),$$

where $\pi(x)$ is the number of finite places v of K with $Nv \le x$. The set S has density equal to $d(S)$ if the limit exists as $x \to \infty$ of $\#\{v \in S|\ Nv \le x\}/\pi(x)$ and is equal to $d(S)$.

Definition 2.1. Suppose Γ is an abstract group and $\rho_i : \Gamma \to GL_n(F)$, for $i = 1, 2$ are two linear representations of Γ. We say that ρ_1 and ρ_2 are *potentially equivalent* if there is a subgroup Γ' of finite index in Γ such that the restriction of ρ_1 and ρ_2 to Γ' are equivalent.

Definition 2.2. Suppose ρ_1 and ρ_2 are two Galois representations of G_K into $GL_n(F)$. We define ρ_1 and ρ_2 to be *locally potentially equivalent* at a set of places T of K, if

for each $v \in T$, the localizations $\rho_{1,v}$ and $\rho_{2,v}$ are potentially equivalent considered as representations of G_{K_v}.

For the representation ρ as above, let G be the algebraic monodromy group attached to ρ over F, i.e., the smallest algebraic subgroup G of M defined over F such that $\rho(G_K) \subset G(F)$. Let G^0 be the identity component of G, and let $\Phi = G/G^0$ be the finite group of connected components of G. Denote by G_i be the algebraic monodromy groups associated to the representations ρ_i for $i = 1, 2$, and let c_i be the number of connected components of G_i for $i = 1, 2$.

One of the main theorems of this note is to observe that two continuous semisimple ℓ-adic representations of a global field K which are locally potentially equivalent at a large enough set of places are in fact potentially equivalent:

Theorem 2.1. *Suppose $\rho_i : G_K \to GL_n(F)$, $i = 1, 2$ are two continuous semisimple ℓ-adic representations of the absolute Galois group G_K of a global field K unramified outside a finite set of places of K, where F is a local field of characteristic zero and residue characteristic ℓ coprime to the characteristic of K. Suppose there exists a set T of finite places of K such that for every $v \in T$, $\rho_{1,v}$ and $\rho_{2,v}$ are potentially equivalent. Assume that the upper density of T,*

$$ud(T) > min\left(1 - \frac{1}{c_1}, 1 - \frac{1}{c_2}\right).$$

Then, ρ_1 and ρ_2 are potentially equivalent, viz. there exists a finite extension L of K such that

$$\rho_1\left.\right|_{G_L} \simeq \rho_2\left.\right|_{G_L}.$$

Corollary 2.1. *With the hypothesis of Theorem 2.1, assume further that one of the algebraic monodromy groups, say G_1 is connected and the representation ρ_1 is absolutely irreducible. Then there exists a Dirichlet character $\chi : G_K \to F^*$ such that*

$$\rho_2 \simeq \rho_1 \times \chi.$$

In particular, if for every $v \in T$ there exists a character χ_v of G_{K_v} such that

$$\rho_{2,v} \simeq \rho_{1,v} \times \chi_v,$$

then there exists a global character η of G_K such that

$$\rho_2 \simeq \rho_1 \times \eta.$$

Remark 2.1. There can be more than one χ_v satisfying $\rho_{2,v} \simeq \rho_{1,v} \times \chi_v$. Hence we cannot expect that the local component of η at v be equal to χ_v.

2.1. **Preliminaries.** For the proof of Theorem 2.1, we recall the main results proved in [Ra1]. First, we recall Theorem 3 of [Ra1], an algebraic interpretation of results proved in Section 6 (especially Proposition 15) of [Se], giving an algebraic formulation of the Chebotarev density theorem for the density of places satisfying an algebraic conjugacy condition:

Theorem 2.2. [Ra1, Theorem 3] *Let M be an algebraic group defined over F. Suppose*

$$\rho : G_K \to M(F)$$

is a continuous representation unramified outside a finite set of places of K.

Suppose X is a closed subscheme of M defined over F and stable under the adjoint action of M on itself. Let

$$C := X(F) \cap \rho(G_K).$$

Let Σ_u denote the set of finite places of K at which ρ is unramified. Then the set

$$S := \{v \in \Sigma_u \mid \rho(\sigma_v) \subset C\}.$$

has a density given by

$$d(S) = \frac{|\Psi|}{|\Phi|},$$

where $\Psi = \{\phi \in \Phi \mid G^\phi \subset X\}$.

Remark 2.2. Since we will be able to compare only the semi-simplifications of the representations, we assume that the representations are semi-simple. In particular, this implies that the algebraic monodromy group is a reductive algebraic group defined over F.

Corollary 2.2. *Let ρ be a semisimple continuous l-adic representation of G_K to $GL_n(F)$ unramified outside a finite set of places of K. Then there is a density one set of places of K at which ρ is unramified and the corresponding Frobenius conjugacy class is semisimple.*

The advantage of the algebraic prescription of positive density is that it allows us to use techniques from the theory of reductive algebraic groups, to change the base field and work over complex numbers. This allows the use of transcendental methods. The following result, essentially proved in Theorem 2 of [Ra1], depends on the fact that the identity matrix is the unique matrix in the unitary group $U(n)$ on n-variables having trace n (see also the proof of Theorem 3.1):

Theorem 2.3. *Suppose ρ_1 and ρ_2 are two semisimple Galois representations of G_K into $GL_n(F)$. Let*

$$\rho := \rho_1 \times \rho_2 : G_K \to GL_n \times GL_n$$

be the direct sum of ρ_1 and ρ_2. Assume that the algebraic monodromy group G_1 attached to ρ_1 over F is connected. Suppose there exists a positive density of unramified finite places T of F for ρ, such that

$$\mathrm{Trace}(\rho_1(\sigma_v)) = \mathrm{Trace}(\rho_2(\sigma_v)) \quad for\ v \in T,$$

where $\rho_1(\sigma_v)$ (resp. $\rho_2(\sigma_v)$) denotes the Frobenius conjugacy class at the place v associated respectively to the representations ρ_1 and ρ_2. Then ρ_1 and ρ_2 are potentially equivalent, i.e., there exists a finite extension L of K such that

$$\rho_1|_{G_L} \simeq \rho_2|_{G_L}.$$

Remark 2.3. Suppose the number of connected components G_1 is c_1. Assume further that the set of places $v \in T$ at which the traces of the two representations are equal has upper density strictly greater than $1 - 1/c_1$. By Theorem 2.2 it follows that there is a connected component of G that surjects onto the connected component of identity in G_1. Arguing as above, we can conclude that the representations are potentially equivalent.

2.2. **Proof of Theorem 2.1.** We first observe the following:

Lemma 2.1. *Let σ_1 and σ_2 be two semisimple elements in $GL_n(F)$ where F is local field (finite extension of $\mathbb{Q}_\ell$). Suppose there exists a non-zero integer k such that σ_1^k and σ_1^k are conjugate in $GL_n(F)$. Then, there exists a positive integer m depending only on n and F such that σ_1^m and σ_2^m are conjugate in $GL_n(F)$.*

Remark 2.4. Since we are working in GL_n, two elements are conjugate in $GL_n(F)$ if and only if they are conjugate in $GL_n(\bar{F})$.

Proof. Choose an algebraic closure $\bar{F}$ of F. Let F' be the extension of F in $\bar{F}$ generated by the eigenvalues of σ_1 and σ_2 in $\bar{F}$. It is easy to see that

$$[F' : F] \leq (n!)^2.$$

The number of roots of unity contained in such a field F' is bounded above by some positive integer m_0 depending only on $[F' : \mathbb{Q}_\ell]$, thus depending only on n and $[F : \mathbb{Q}_\ell]$.

Let $\{\alpha_1, \cdots, \alpha_n\}$ (respectively $\{\beta_1, \cdots, \beta_n\}$) be the eigenvalues of σ_1 (respectively σ_2). Since by our hypothesis σ_1^k is conjugate of σ_2^k we have up to a permutation,

$$\alpha_i^k = \beta_i^k, \qquad \forall\; 1 \leq i \leq n.$$

Hence α_i and β_i differ by a root of unity, which lies in F'. Thus from the above comment, for $m = m_0!$ we have:

$$\alpha_i^m = \beta_i^m, \qquad \forall\; 1 \leq i \leq n.$$

But since both σ_1 and σ_2 are semisimple elements in $GL_n(\bar{F})$, σ_1^m and σ_2^m are conjugate in $GL_n(F)$. $\square$

Lemma 2.2. *Let L/K be a Galois extension of degree d. Suppose T is a set of places of K of upper density $\delta > 1 - 1/d$. Then the set of places T' of L lying above places in T is of positive upper density.*

Proof. Let the upper density of T be δ. By Chebotarev density theorem and hypothesis, there is a subset T_L of places of K with upper density $\delta - (1 - 1/d)$ such that every place $v \in T_L$ splits completely in L. Then the set of places T' of L lying above a place in T_L is of positive upper density $d(\delta - (1 - 1/d))$. $\square$

We now give the proof of Theorem 2.1.

Proof of Theorem 2.1. We first observe that by Corollary 2.2, we can assume that T consists of places v of K at which both ρ_1 and ρ_2 are unramified and the corresponding Frobenius conjugacy classes are semisimple.

Let $\rho := \rho_1 \times \rho_2$. Let G, G_1, and G_2 be respectively the algebraic monodromy groups of ρ, ρ_1, and ρ_2. By renaming, we can assume that $c_1 \leq c_2$. Let G^0 and G_1^0 be respectively the connected components of identity of G and G_1. Let $\Phi := G/G^0$ be the group of connected components of G. For every $\phi \in \Phi$, denote by G^ϕ be the corresponding connected component.

Let L_1 be the finite extension of K obtained by taking the invariant field of the kernel of the homomorphism $G_K \to G_1/G_1^0$. The degree of L_1 over K is precisely c_1. Let T' be the set of places of L_1 lying above a place in T of K. By Lemma 2.2 and the hypothesis, the upper density of T' is positive.

Further the algebraic monodromy group of $\rho_1|_{G_{L_1}}$ is connected, given by the connected component G_1^0 of G_1. Hence we have reduced to the case that G_1 is connected.

For every $v \in T'$, it follows that there exists an integer n'_v divisible by n_v (we can assume $n_v = n'_v$ by working with v of degree one over K) such that $\rho_1|_{G_{L_1}}(\sigma_v{}^{n_v})$ and $\rho_2|_{G_{L_1}}(\sigma_v{}^{n_v})$ are conjugate in $GL_n(F)$.

By Lemma 2.1, there exists a positive integer m independent of $v \in T'$, and such that for all $v \in T'$, $\rho_1(\sigma_v{}^m)$ and $\rho_2(\sigma_v{}^m)$ are conjugate in $GL_n(F)$. In particular, we have

$$\rho(\sigma_v) \in X_m, \quad \forall v \in T',$$

where

$$X_m := \{(g_1, g_2) \in GL_n \times GL_n \mid \mathrm{Trace}(\mathrm{g}_1^\mathrm{m}) = \mathrm{Trace}(\mathrm{g}_2^\mathrm{m}) \}.$$

Now X_m is a Zariski closed subvariety of $GL_n \times GL_n$ invariant under conjugation.

Since T' is of positive upper density, by Theorem 2.2 above, there exists a connected component G^ϕ of G such that G^ϕ is contained in X_m.

Consider the map $x \mapsto x^m$ from G to itself. Under this map, G^ϕ maps *onto* a connected component G^ψ of G. Since G^ϕ is contained inside X_m, G^ψ is contained inside X_1.

We now argue as in the proof of Theorem 2 of [Ra1] (Theorem 2.3 as above) to conclude that there exists a finite extension L of K such that

$$\rho_1|_{G_L} \simeq \rho_2|_{G_L}.$$

$\square$

Remark 2.5. If we assume that $ud(T) = 1$, then a simpler proof can be given: we first appeal to Lemma 2.1 and Theorem 2.2 to conclude that $G \subset X_m$. Since the map $x \mapsto x^m$ is surjective from G^0 to itself, we get that $G^0 \subset X_1$. The theorem then follows from the Chebotarev density theorem.

2.3. **Proof of Corollary 2.1.** For the proof of Corollary 2.1, we recall the following proposition proved in ([Ra1, Ra2]):

Proposition 2.4. *Suppose ρ_1 and ρ_2 are two Galois representations of G_K into $GL_n(F)$ satisfying the following:*

- *There exists a finite extension L of K such that*
$$\rho_1|_{G_L} \simeq \rho_2|_{G_L}.$$
- *The representation $\rho_1|_{G_L}$ is absolutely irreducible (this is guaranteed if we know that the algebraic monodromy group G of ρ is connected and the representation ρ of G is absolutely irreducible).*

Then there exists a finite order character $\chi : G_K \to F^*$ such that
$$\rho_2 \simeq \rho_1 \otimes \chi.$$

The proposition follows from Schur's lemma by standard arguments. Combining the above proposition with Theorem 2.1, yields a proof of Corollary 2.1.

2.4. **Example: Tensor and Symmetric powers.** We now give examples where an algebraic relation between the representations forces potential equivalence. For a linear representation ρ of a group Γ, denote by χ_ρ it's character. For a positive integer k, let $T^k(\rho)$ (resp. $S^k(\rho)$) denote the k-th tensor (resp. symmetric) power representation associated to ρ. The following theorem is proved in [Ra2]:

Theorem 2.5. *Let Γ be an abstract group and $\rho_i : \Gamma \to GL_n(F)$ for $i = 1, 2$ be two semisimple representations of Γ, where F is a field of characteristic zero. Suppose that for some natural number k, either the k-th tensor power or the symmetric power representations of ρ_1 and ρ_2 become isomorphic.*

Then ρ_1 and ρ_2 are potentially isomorphic.

Proof. For the sake of completeness and also serves to illustrate the power of the algebraic method, we give an outline of the proof. We replace Γ by the algebraic envelope G in $GL_n \times GL_n$ of the image of Γ inside $(GL_n \times GL_n)(F)$. The algebraic envelope is a reductive group, and by going to a subgroup of finite index one can assume that G is connected. We can further take F to be the field of complex numbers.

For the tensor power the hypothesis implies that the k-th powers of the characters are equal on G. Hence the characters of the representations differ by a k-th root of unity. Since the characters are equal at identity, they are equal in a connected neighbourhood of identity. Since a neighbourhood of identity is Zariski dense in a connected algebraic group, and the characters are regular functions on the group, it follows that the characters are equal on G.

For the symmetric powers, the proof proceeds by imposing a lexicographic total ordering on the set of weights of a compact maximal torus D contained in G. This is

possible since the weights are totally imaginary on the Lie algebra $\mathrm{Lie}(D)$ of D, and we can use the ordering on the reals. If

$$\lambda_1 \geq \cdots \geq \lambda_n \quad (\text{resp. } \mu_1 \geq \cdots \mu_n$$

are the weights of the representation ρ_1 (resp. ρ_2) of $\mathrm{Lie}(D)$ in $\mathrm{End}(\mathbb{C}^n)$, the leading n weights of the k-th symmetric power are given by

$$k\lambda_1 \geq (k-1)\lambda_1 + \lambda_2 \cdots \geq (k-1)\lambda_1 + \lambda_n \geq \cdots$$

$$\text{resp.} \quad k\mu_1 \geq (k-1)\mu_1 + \mu_2 \cdots \geq (k-1)\mu_1 + \mu_n \geq \cdots$$

Arguing inductively allows us to say that the set of weights occurring in the two representations of G are equal, and hence ρ_1 and ρ_2 are potentially isomorphic. $\square$

Combining the above theorem with Theorem 2.1, we have the following corollary:

Corollary 2.3. *Suppose $\rho_i : G_K \to GL_n(F)$, $i = 1, 2$ are continuous semisimple ℓ-adic representations of the absolute Galois group G_K of a global field K. Suppose there exists a set T of finite places of K with $ud(T) > min(1 - 1/c_1, 1 - 1/c_2)$ such that*

$$R_v \circ \rho_{1,v} \simeq R_v \circ \rho_{2,v},$$

where $R_v = T^{n_v}$ or S^{n_v} is either a n_v-th tensor or symmetric power representation for some natural number n_v, n_v depending on v. Then there is a finite extension L of K such that

$$\rho_1 \mid_{G_L} \simeq \rho_2 \mid_{G_L} .$$

3. Powers of character values of Frobenius classes

In Theorem 2.1, we worked with equality of conjugacy classes. We now formulate a result that will work with character values.

Theorem 3.1. *With notation as in Theorem 2.1, assume that there exists a set T of finite places of K such that for every $v \in T$, there exists non-zero integers $m_v > 0$ satisfying the following:*

$$\chi_{\rho_1}(\sigma_v)^{m_v} = \chi_{\rho_2}(\sigma_v)^{m_v}$$

Assume that the upper density of T satisfies the inequality,

$$ud(T) > min \left(1 - \frac{1}{c_1}, 1 - \frac{1}{c_2} \right),$$

Then, ρ_1 and ρ_2 are potentially equivalent, viz. there exists a finite extension L of K such that

$$\rho_1 \mid_{G_L} \simeq \rho_2 \mid_{G_L} .$$

For a fixed place, the hypothesis is on a single power of the trace of the conjugacy class, and thus does not necessarily imply local equivalence. However the global arguments as in the proof of the above theorem helps us in proving this theorem.

Proof. Arguing as in the proof of the Theorem 2.1, we can assume that G_1 is connected. If $\chi_{\rho_1}(\sigma_v)$ vanishes, then the hypohesis holds for any integer m_v. On the other hand, if $\chi_{\rho_1}(\sigma_v)$ is non-zero, then $\chi_{\rho_1}(\sigma_v)$ and $\chi_{\rho_2}(\sigma_v)$ differ by a root of unity belonging to F. Since the group of roots of unity in the non-archimedean local field F is finite, there is an integer m independent of v, such that for $v \in T$,

$$\chi_{\rho_1}(\sigma_v)^m = \chi_{\rho_2}(\sigma_v)^m$$

Let

$$X^m := \{(g_1, g_2) \in GL_n \times GL_n \mid \mathrm{Trace}(g_1)^m = \mathrm{Trace}(g_2)^m \}.$$

X^m is a Zariski closed subvariety of $GL_n \times GL_n$ invariant under conjugation. By Theorem 2.2, the density condition on T implies the existence a connected component $G^\phi \subset X^m$. By working over the complex numbers and with a maximal compact subgroup J of G, we can assume that there is an element of the form $(1, y) \in J^\phi \cap X^m$. Since the only elements in an unitary group $U(n)$ with the absolute value of it's trace being precisely n are scalar matrices ζI_n with $|\zeta| = 1$, we conclude that y is of the form ζI_n for ζ a m-th root of unity.

We can write the connected component $G^\phi = G^0.(1, \zeta\, I_n)$. In particular, every element $(u_1, u_2) \in G^0$ the identity component of G, can be written as

$$(u_1, u_2) = (z_1, \zeta^{-1} z_2),$$

where $(z_1, z_2) \in G^\phi \cap X_m$. Since ζ is a m-th root of unity, we have

$$\mathrm{Trace}(u_1^m) = \mathrm{Trace}(z_1^m) = \mathrm{Trace}(z_2^m) = \mathrm{Trace}((\zeta^{-1} z_2)^m) = \mathrm{Trace}(u_2^m).$$

Hence $G^0 \subset X^m$.

We are now in the situation of Theorem 2.5. Let p_i, $i = 1,\ 2$ be the two projections from G^0 to $GL(n)$. The statement $G^0 \subset X^m$ can be reformulated as saying that

$$\chi_{p_1}^m = \chi_{p_2}^m,$$

restricted to G^0. Since G^0 is connected, it follows from Theorem 2.5, that the representations p_1 and p_2 are equivalent restricted to G^0. Hence it follows that ρ_1 and ρ_2 are potentially eqiuvalent.

$\square$

Remark 3.1. If we just assume that $ud(T) = 1$, then the proof can be given as follows: we appeal to Theorem 2.2, base change to the extension L as in the proof of the theorem and then invoke Theorem 2.5 to complete the proof.

Again, combining with Proposition 2.4, we have the following corollary:

Corollary 3.1. *Suppose that the algebraic monodromy group G_1 is connected and the representation ρ_1 is absolutely irreducible. Assume that there exists a set of unramified places T for ρ of positive density and a collection of positive integers m_v for $v \in T$, such that*

$$\chi_{\rho_1}(\sigma_v)^{m_v} = \chi_{\rho_2}(\sigma_v)^{m_v} \qquad \forall v \in T.$$

Then there exists a Dirichlet character $\chi : G_K \to F^*$ *such that*

$$\rho_2 \simeq \rho_1 \times \chi.$$

4. AN ALGEBRAIC DENSITY RESULT

We have the following density result, analogous to the Chebotarev density theorem that the Frobenius conjugacy classes are Zariski dense in the algebraic monodromy group:

Theorem 4.1. *Let* $\rho : G_K \to GL_n(F)$ *be a continuous semisimple ℓ-adic representation with connected algebraic monodromy group* G. *Let* T *be a set of unramified places for ρ with positive upper density. For $v \in T$, let n_v be a positive integer. Then, the smallest algebraic subgroup of G generated by $\rho(\sigma_v)^{n_v}$, where σ_v is the Frobenius conjugacy class associated to v, as v ranges over T is G.*

Proof. Since ρ is semisimple, G is a reductive algebraic subgroup of GL_n over F. Let G_T be the smallest algebraic subgroup of G generated by $\rho(\sigma_v)^{n_v}$, where σ_v is the Frobenius conjugacy class associated to v, as v ranges over T. Then G_T is a closed, normal subgroup of G. Choose an embedding of G/G_T as an algebraic subgroup of GL_m over F for some positive integer m. Denote by ρ_1 the map $\rho : G_K \to G$ followed by quotient to the reductive algebraic group $G/G_T \subset GL_m$. Let $\rho_2 = Id : G_K \to GL_m$ be the *identity* map sending every element of G_K to the identity element of GL_m. Applying Theorem 2.1 to ρ_1 and ρ_2, it follows that ρ_1 and ρ_2 are potentially equivalent. Thus, for some finite extension L of K,

$$\rho_1|_{G_L} \simeq Id|_{G_L}.$$

Thus, $\rho_1(G_L) = \{1\}$. But since $\rho_1(G_K)$ is Zariski dense in the connected algebraic group G/G_T, we get $G/G_T = \{1\}$. $\qquad\square$

Remark 4.1. It is tempting to conjecture that we can combine Theorems 2.1 and 3.1 into one single theorem, viz., to be able to conclude from an equality of character values of the form

$$\chi_{\rho_1}(\sigma_v^{m_v})^{k_v} = \chi_{\rho_2}(\sigma_v^{m_v})^{k_v} \qquad \forall v \in T.$$

at a sufficiently large set of places T, that χ_1 and χ_2 are potentially equivalent. However this would require in the statement of the foregoing corollary, that the collection of conjugacy classes $\{\rho(\sigma_v)^{n_v}\}$ is Zariski dense in G. It is possible that such a statement holds in the context of l-adic representations, but this is not a general statement about Zariski dense subsets in connected groups. For example, the set $\{\sigma_n = 1/n\}$ is Zariski dense in the affine line G_a, but if we consider arbitrary multiples $\{n\sigma_n = 1\}$ then it is no longer Zariski dense in G_a.

5. Modular forms

We now indicate briefly the proof of Theorem 1.1 stated in the introduction. For a newform f of weight k, let

$$\rho_f : G_{\mathbb{Q}} \to GL_2(\mathbb{Q}_l),$$

be the l-adic representations of $G_{\mathbb{Q}}$ associated by the work of Shimura, Ihara and Deligne. It has been shown by Ribet in [Ri], that the representation ρ_f is semisimple. Further if f is a non-CM form of weight at least two, then the Zariski closure G_f of the image $\rho(G_{\mathbb{Q}})$ is GL_2. Theorem 1.1 follows now from Corollary 3.1.

Remark 5.1. A similar statement can be made for the class of Hilbert modular forms too.

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